Non-Convex MIQCP in Gurobi 9.1: New Advances

GUROBI OPTIMIZATION

The World's Fastest Solver

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Non-Convex MIQCP Performance

• **Presolve**		**33%**
• Detect parallel Q constraints	4%	
• Add Q equations to Q constraints to cancel quadratic terms	18%	
• Add Q equations to objective to cancel quadratic terms	0%	
• Improved bilinear probing code	3%	
• Accept small bound changes for variables that appear in quadratic terms	2%	
• Allow substitutions on variables that appear in linear part of Q constraints	1%	
• Feasibility-based bound tightening on variables in quadratic terms	2%	
• **QCP to Bilinear Translation**		**36%**
• Also disaggregate Q constraints with only positive squares plus linear terms	8%	
• Convert positive squares of objective into constraint	14%	
• Clean up translation code to save some work	1%	
• Reuse product variables across bilinear and convex Q constraints	9%	
• **Node presolve**		**4%**
• Tighten finite bound for variables with one infinite bound	1%	
• Re-propagate bilinear constraints if domain of mixed product term changed	1%	
• Faster propagation for disjoint product terms	1%	
• Exploit implied quadratic equations in propagation	1%	

Time limit: 10000 seconds
Speed-ups on 217 models that take at least 1 second

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MINLP Performance – Summary

• **Branching**		**15%**
• Adjust balance of McCormick volume and violation scores	15%	
• **Cuts**		**22%**
• Tangent cuts for convex part of bilinear constraints	1%	
• Tilt tangent cuts to increase Euclidean violation	1%	
• Exploit implied quadratic equations in cuts	4%	
• PSD cuts	15%	
• **Primal Heuristics**		**2%**
• Randomize order for greedy Q term coverage in fix-and-dive	1%	
• Consider quadratic constraints in a sub-MIP heuristic	1%	
• **Simplex/MIP Integration**		**2%**
• Add bias to favor moving McCormick constraints into basis	2%	
• **Other Improvements**		**50%**
• Including effects of MIP/LP/QP/QCP improvements		
• **Total**		**4.11x**

Time limit: 10000 seconds
Speed-ups on 217 models that take at least 1 second

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Parallel Quadratic Constraints

- **Identify quadratic constraints that are parallel to each other**
  - Example from customer model:
```
  0.259286x155 + ... + 0.259286x7563 - x18079 + [ +2x18078*x18079 ] <=  1
- 0.259286x155 - ... - 0.259286x7563 + x18079 + [ -2x18078*x18079 ] <= -1
```
  - Can be merged into equation:
```
  0.259286x155 + ... + 0.259286x7563 - x18079 + [ +2x18078*x18079 ] ==  1
```
  - Other case: discard identical or dominated constraint
- **Happens frequently in sub-MIPs solved by primal heuristics**
- **Detection is very similar to linear case**
  - Hash function for linear and quadratic parts (normalize for sign/scaling)
  - Pairwise comparison of constraints with identical hash value
  - Very fast in practice

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Parallel Quadratic Constraints

- Affects about 20% of models in non-convex MIQCP test set
  - 11% speed-up on those models
  - 4% speed-up overall
  - 12 consistent wins, 0 consistent losses

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Substitute Identical Quadratic Part

- **Different linear part, but identical quadratic part**
- **Case 1**
  - At least one constraint is an equation

$$\begin{gathered} a^1x + x^TQx = b^1 \\ a^2x + x^TQx \leq b^2 \end{gathered}$$

  - Subtract equation from other constraint turns other into linear constraint

$$\begin{gathered} a^1x + x^TQx = b^1 \\ (a^2 - a^1)x \leq b^2 - b^1 \end{gathered}$$

- **Case 2**
  - Both constraints are inequalities

$$\begin{gathered} a^1x + x^TQx \leq b^1 \\ a^2x + x^TQx \leq b^2 \end{gathered}$$

  - Introduce auxiliary variable to represent quadratic part

$$\begin{gathered} a^1x + s \leq b^1 \\ a^2x + s \leq b^2 \\ x^TQx - s \leq 0 \end{gathered}$$

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Substitute Identical Quadratic Part

- Affects about 27% of models in non-convex MIQCP test set
  - 39% speed-up on those models
  - 18% speed-up overall
  - 22 consistent wins, 0 consistent losses

- Reducing Q part of objective affects only 7 models (< 2%)
  - 5% speed-up on those models
  - 0.1% speed-up overall
  - 0 consistent wins, 0 consistent losses

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PSD Cuts

- **New cutting plane separator in Gurobi 9.1 for non-convex MIQCPs**
  - Controlled by PSDCuts parameter
- **Sherali and Fraticelli (2002):**
  - "Enhancing RLT relaxations via a new class of semidefinite cuts"
- **Qualizza, Belotti and Margot (2012):**
  - "Linear Programming Relaxations of Quadratically Constrained Quadratic Programs"

- **Bilinear constraints:** $y_{ij} = x_i x_j$
- **As matrix equation:** $Y = xx^T$
- **Relax to:** $Y \succcurlyeq xx^T \Leftrightarrow Y - xx^T \succcurlyeq 0$ **(matrix is PSD)**
- **Schur's complement:** $Y - xx^T \succcurlyeq 0 \Leftrightarrow \begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} \succcurlyeq 0$
- **Equivalent to:** $v^T \begin{pmatrix} 1 & x^T \\ x & Y \end{pmatrix} v \geq 0$ **for all** $v \in \mathbb{R}^{n+1}$
- **Separate cuts by finding v for which this is violated**
  - Eigenvectors corresponding to negative eigenvalues

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PSD Cuts

- **We separate PSD cuts for up to 10 product variables**
  - Only use those sets where all mixed y_{ij} variables exist
    - Find cliques in graph with nodes x_j and edges y_{ij}
- **Example for single mixed product variable y_{ij}**
  - Find v with $v^T \begin{pmatrix} 1 & x_i^* & x_j^* \\ x_i^* & y_{ii}^* & y_{ij}^* \\ x_j^* & y_{ij}^* & y_{jj}^* \end{pmatrix} v < 0$ for current LP solution (x^*, y^*)
    - Find negative eigenvalue, let v be the corresponding eigenvector
  - Add cut $v^T \begin{pmatrix} 1 & x_i & x_j \\ x_i & y_{ii} & y_{ij} \\ x_j & y_{ij} & y_{jj} \end{pmatrix} v =$

$$v_1^2 + 2v_1v_2x_i + 2v_1v_3x_j + v_2^2y_{ii} + v_3^2y_{jj} + 2v_2v_3y_{ij} \geq 0$$

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PSD Cuts

- **Affects about 34% of models in non-convex MIQCP test set**
  - 32% speed-up on those models
  - 15% speed-up overall
  - 28 consistent wins, 5 consistent losses

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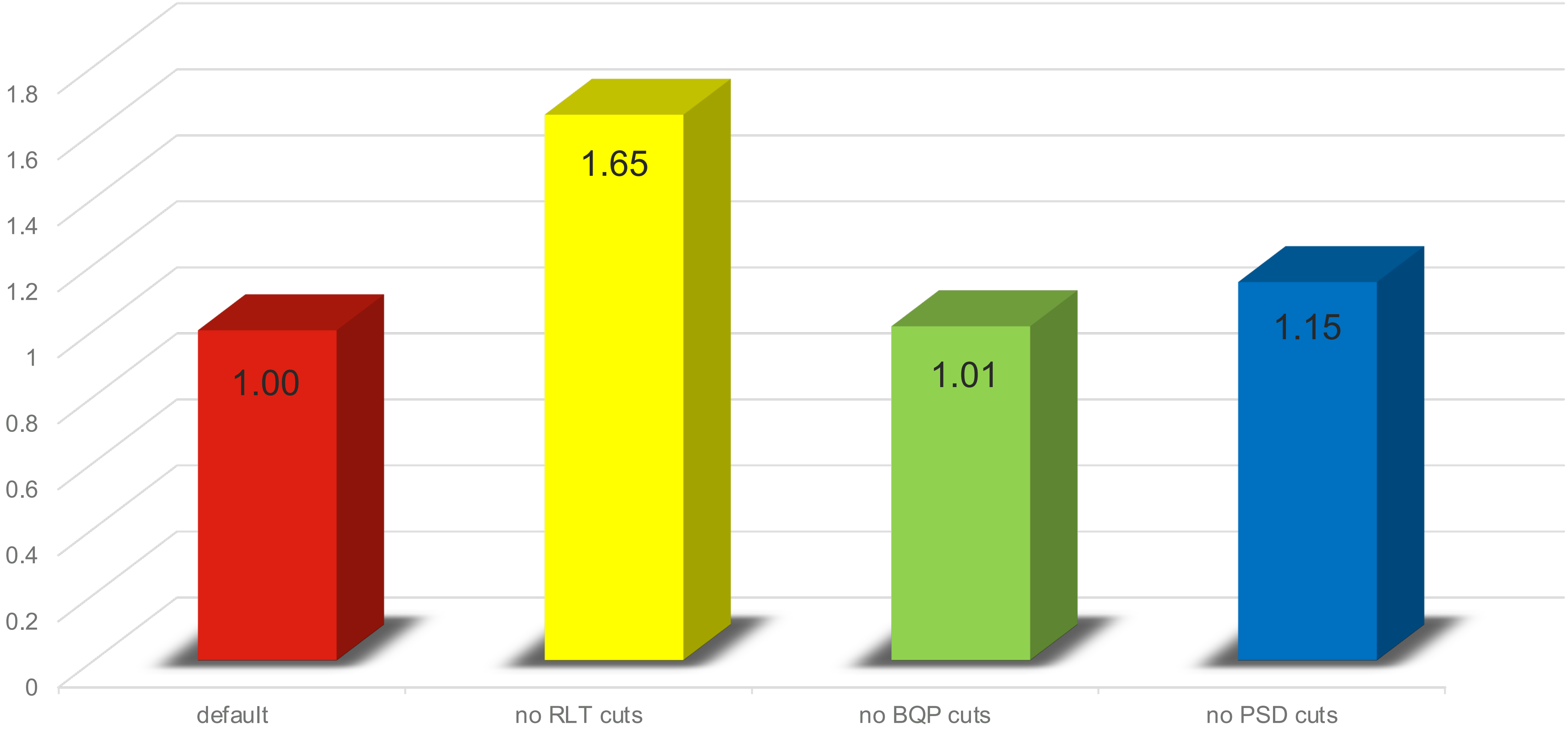
Non-Convex MINLP Cuts Summary



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Bilinear Inequalities and Cuts

- **Recall PSD cuts, formulated with y_{ij} variables**
  - $v^T \begin{pmatrix} 1 & x_i & x_j \\ x_i & y_{ii} & y_{ij} \\ x_j & y_{ij} & y_{jj} \end{pmatrix} v =$

$$v_1^2 + 2v_1v_2x_i + 2v_1v_3x_j + v_2^2y_{ii} + v_3^2y_{jj} + 2v_2v_3y_{ij} \geq 0$$

- **But actually, one can view this as a two step process**
  - Formulate quadratic cut in x_j variables
    - $v_1^2 + 2v_1v_2x_i + 2v_1v_3x_j + v_2^2x_i^2 + v_3^2x_j^2 + 2v_2v_3x_ix_j \geq 0$
  - Substitute quadratic terms for y variables using $y_{ij} = x_ix_j$
  - But what if we only have $y_{ij} \leq x_ix_j$ or $y_{ij} \geq x_ix_j$?

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Bilinear Inequalities and Cuts

- **Same question for RLT and PSD cuts**
  - Given a quadratic cut
    $$\sum a_{ij} x_i x_j \leq b$$
    and relations
    $$y_{ij} = x_i x_j,\ y_{ij} \leq x_i x_j, \text{ or } y_{ij} \geq x_i x_j$$
    how can we derive a valid linear cut?
- **Need to look at signs of a_{ij} coefficients**
  - $a_{ij} > 0$: can only use $y_{ij} = x_i x_j$ and $y_{ij} \leq x_i x_j$
  - $a_{ij} < 0$: can only use $y_{ij} = x_i x_j$ and $y_{ij} \geq x_i x_j$
  - If not compatible: need to relax term
    - E.g., by substituting bounds for x_i and x_j that minimize $a_{ij} x_i x_j$
- **Observation: bilinear equations help to find better cuts**

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Implied Quadratic Equations

- Consider a quadratic inequality

$$a'x' + ax + y^T Qy \leq b$$

  with
  - the linear part partitioned into ax and $a'x'$, and
  - the set of variables in the quadratic part being disjoint from the linear part
- The inequality is an implied equation if
  - for any x' and y we can always move ax upwards until we hit b, or
  - for any x' and x we can always move $y^T Qy$ upwards until we hit b.

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Implied Quadratic Equations

- **Exploit implied quadratic equations in**
  - Cuts
    - Allows more substitutions of bilinear terms by product variables
    - RLT cuts
    - PSD cuts
    - BQP cuts
  - Propagation
    - Propagate constraint in opposite direction
    - Node presolve
    - Fix-and-dive heuristics
  - Branching
    - Update shadow costs of variable for both directions

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Implied Quadratic Equations

- **Affects about 32% of models in non-convex MIQCP test set**
  - 9% speed-up on those models
  - 4% speed-up overall
  - 8 consistent wins, 1 consistent loss

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Non-Convex MIQCP Performance Gurobi 9.0 vs. 9.1

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Total run-time over all 1524 models in log files: 7282968 sec = 2023.0 h = 84.3 d

Full set	Count	Loss/Win	NodeR	IterR	VMemR	TimeR	
all:	729	52/ 154	0.577	0.618	0.847	0.657	
>0s:	413	52/ 154	0.363	0.433	0.739	0.442	
>1s:	217	47/ 140	0.169	0.226	0.649	0.245	4.1x speed-up
>10s:	161	31/ 112	0.111	0.156	0.583	0.158	
>100s:	123	19/ 94	0.073	0.116	0.517	0.104	9.6x speed-up
>1000s:	85	14/ 66	0.062	0.103	0.462	0.081	

Unsolved:	(31 / 4)	+ 316 for all solvers
- Time limit:	(34 / 8)	+ 307 for all solvers
- Mem limit:	(4 / 3)	+ 2 for all solvers
No feasible:	(14 / 4)	+ 111 for all solvers

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Non-Convex MIQCP Performance Other Solvers vs. Gurobi 9.1

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- Comparison of other solvers vs. Gurobi 9.1 conducted by Prof. Hans Mittelmann on models from QPLIB
  - See http://plato.asu.edu/bench.html
  - Gurobi 9.0 results from 8 October 2020 (discrete non-convex) and 10 October 2020 (continuous non-convex)
  - Gurobi 9.1 results from 10 November 2020 (discrete non-convex) and 2 December 2020 (continuous non-convex)
  - Antigone, BARON, FSCIP, Couenne, Minotaur, SCIP, Octeract, Gurobi
- Binary Non-Convex QPLIB Benchmark
  - Not relevant here: translate into MILP
- Convex Continuous QPLIB Benchmark
  - Not relevant here: these are convex SOCPs
- Convex Discrete QPLIB Benchmark
  - Not relevant here: these are convex MIQCPs

Problem Class	#	Gurobi 9.0 solved	Gurobi 9.1 solved	Best Competitor	Competitor solved	Competitor vs. Gurobi 9.0	Competitor vs. Gurobi 9.1
Continuous non-convex	57	28	35	Antigone	29	1.59x	4.68x
Discrete non-convex	75	65	66	FSCIP	32	7.31x	10.5x

Solved by at least one solver

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Thank You!

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Implied Quadratic Equations

- **Quadratic inequality** $a'x' + ax + y^T Q y \leq b$
- **Conditions for being able to move** ax **upwards:**
  - $\inf\{a'x'\} + \sup\{ax\} + \inf\{y^T Q y\} \geq b$
  - For all integer feasible (x', y) there exists integer feasible x such that
    $$ax + a'x' + y^T Q y = b$$
  - None of the x_j appear in equations
  - Each of the x_j appears in other inequalities only with opposite sign
    - $a_j > 0 \Rightarrow A_{ij} \leq 0$ for all other constraints i
    - $a_j < 0 \Rightarrow A_{ij} \geq 0$ for all other constraints i
  - Similar for objective function
    - $a_j > 0 \Rightarrow c_j \leq 0$
    - $a_j < 0 \Rightarrow c_j \geq 0$
  - Similar for other quadratic constraints and SOS constraints
- **Similar conditions for being able to move** $y^T Q y$ **upwards**

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